Third Midterm Review Solutions

Problem 1:
Suppose we have a pair of trick coins as follows. The first coin is fair, but if the first coin shows heads then the second coin automatically will as well, and if the first coin shows tails then the second coin is fair.
1) Compute expected value and covariance matrix of the two random variables.
2) Find linear combinations of the two random variables which are uncorrelated, and compute their variances.

Problem 2:
Consider the data set given by the following points in the xy plane:

$$\{1,1\},\{2,3\},\{3,5\},\{4,7\}$$

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(1) Compute the covariance matrix by applying singular value decomposition.
(2) Find linear combinations of the variables which are uncorrelated and their variances.

Problem 3:
Consider the matrix

$$A = \begin{bmatrix} 5 & 0 & 1 \\ 0 & 5 & 2 \end{bmatrix}$$

(1) Compute the Singular Value Decomposition of A.
(2) Write A as a sum of rank 1 matrices.
(3) Compute the Pseudo-Inverse of A.
(4) Find a w such that Aw is closest to $\boldsymbol{b} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$. Then compute the projection of $\boldsymbol{b}$ onto $C(A)$.

Problem 4:
1) Find an orthonormal basis of $\mathbb{R}^2$ in which the matrix

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

is diagonal. Is this matrix positive (semi)-definite?
2) Is the matrix

$$\begin{bmatrix} 3 & 7 \\ 7 & -1 \end{bmatrix}$$

positive (semi)-definite?